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INFLUENCE OF THE INCLINATION OF THE SEA FLOOR (SLOPE) ON INTRUSION IN LITTORAL ZONES

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Consideration is given to the two-dimensional (in the vertical plane) steady-state flow of fresh groundwater in a semiinfinite pressure water-bearing layer to a saltwater sea the littoral part of the zone of whose floor makes an arbitrary angle with the horizon. An algorithm of calculation of the intrusion (i.e., penetration) of seawater into the littoral freshwater strata has been developed based on this model. A hydrodynamic analysis of the structure and characteristic features of the process modeled is given using the exact analytical dependences obtained with the P. Ya. Polubarinova-Kochina method; the influence of each determining physical parameter of the model and of the angle of inclination of the littoral part of the sea-floor zone on the degree of intrusion has been evaluated.

Introduction. It is usually assumed in problems of intrusion of seawater into littoral water-bearing strata that the sea floor is horizontal [1] or the outflow of freshwater in littoral zones occurs to horizontal drainage slits [2]. Investigation [3] where the boundary between freshwater and saltwater was interpreted for the first time as the analog of the seepage gap and the vertical position of this boundary was considered is a substantial elaboration of these ideas. This work became the starting point in further investigations of such problems [4–7].

Below, we study the most general case where the motion of freshwater occurs to a sea the littoral part of the zone of whose floor (slope) makes an arbitrary angle $\pi\nu (0 < \nu \le 1)$ with the horizon. To solve this problem we use the Polubarinova-Kochina method [8] which is based on the use of the analytical theory of Fuks linear differential equations [9]. The general solution is represented in parametric form as integrals of hypergeometric functions and irrational factors and is quite analogous to the well-known Polubarinova-Kochina solution of the problem on filtration of groundwater in a trapezoidal bridge in the absence of evaporation [8]. Thereafter the solution is reduced to a form convenient for computations. Finally, the influence of each physical parameter of the model and of the angle of inclination of the littoral part of the sea-floor zone on the character and degree of intrusion is analyzed using the exact analytical dependences obtained and numerical calculations.

It emerges that the case of vertical inclination $\nu = 0.5$ (Mikhailov scheme [3]) is the boundary: when $0 < \nu < 0.5$ the velocity hodograph represents a bounded region, whereas in the case $0.5 \le \nu \le 1$ it represents an unbounded region; a known modular triangle with zero angles is obtained for $\nu = 0.5$ [8, 9], with the result that the transition from one case ($0 < \nu < 0.5$) to another ($0.5 \le \nu \le 1$) is accompanied by the transformation of the velocity-hodograph region and by the reconstruction of the solution. Nonetheless, the final geometric and filtration characteristics remain the same for both cases.

Solutions for the Mikhailov scheme ($\nu = 0.5$) follow as particular cases from the general solution; the well-known Polubarinova-Kochina solution [8] of the problem on filtration of groundwater in an infinitely wide bridge and the Bear–Dagan scheme ($\nu = 1$) which is limiting for the initial model are given within the framework of this solution. In the first case the hypergeometric functions appearing in the solution degenerate into complete elliptic integrals of the first kind, whereas in the second case they degenerate into elementary functions.

Formulation of the Problem and Its Solution. *Case* $0 < \nu < 0.5$. Freshwater of density ρ_1, moving in a semi-infinite pressure water-bearing horizontal stratum enters the sea with a denser salt water of density ρ_2 ($\rho_2 > \rho_1$); the littoral part of the sea-floor zone makes an arbitrary angle $\pi\nu (0 < \nu \le 1)$ with the horizon. As a result, the initially in-

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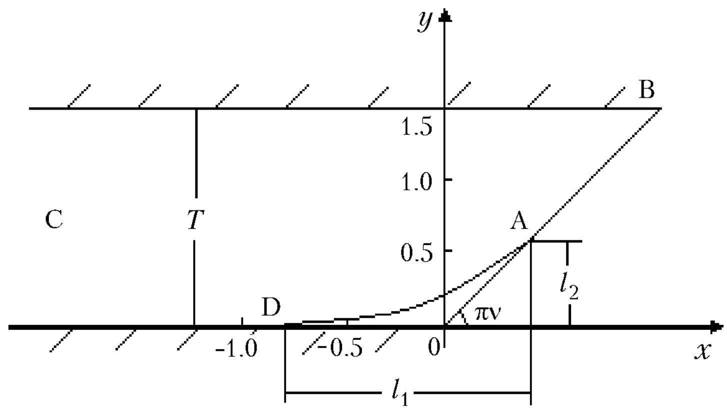

Fig. 1. Flow pattern calculated for $\nu = 0.25$, $T = 1.5$, $Q = 0.007$, and $\rho = 0.01$.

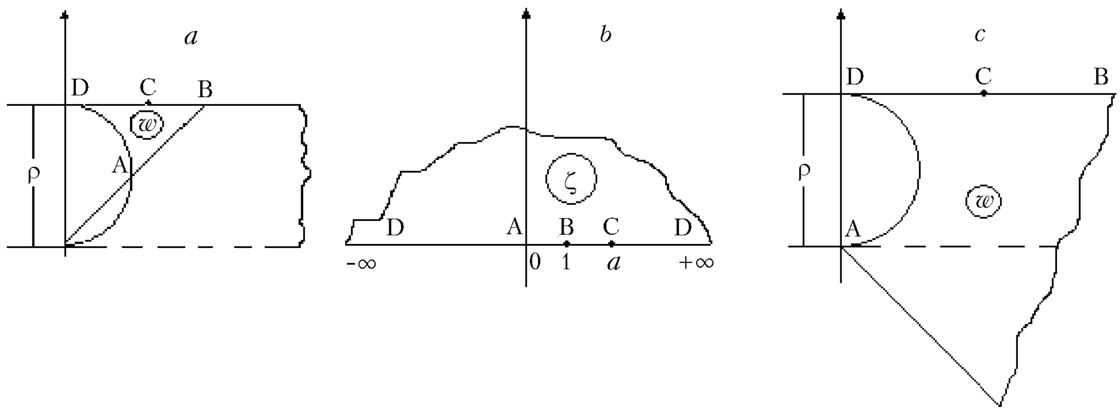

Fig. 2. Regions of complex velocity for the flow diagram for $0 < \nu < 0.5$ (a), auxiliary parametric variable (b), and complex velocity for the flow diagram for ν for $0.5 \le \nu \le 1$ (c).

clined boundary between the moving freshwater and the quiescent saltwater begins to lose its shape in the right-hand lower part of the stratum, shifting to the left toward the flow. After a time, we can have steady-state motion [10, 11] where the brine settles down, the boundary turns out to be the streamline for freshwater, and the motion of saltwater toward the ground takes the shape of a tongue penetrating into the freshwater stratum (Fig. 1).

We will assume that the motion of groundwater obeys Darcy's law with a known filtration factor κ = const and occurs in a homogeneous isotropic ground which is considered to be incompressible, just as the liquid filtering through it. The thickness of the water-bearing layer T, the filtration flow rate Q, and the parameter $\rho = \frac{\rho_2}{\rho_1} - 1$ are assumed to be prescribed. The influence of capillary and diffusion phenomena at the boundary is disregarded, as is customary in such kinds of problems [1–3, 10, 11].

Under such assumptions traditional for the class of flows in question, mathematical modeling of the process studied is reduced to determination of the complex flow potential $\omega(z)$ with the following boundary conditions:

$$\text{AB:}\ \ \varphi = \rho y\,,\ \ y = x\tan\pi\nu\,;\ \ \text{BC:}\ \ \phi = Q\,,\ \ y = T\,;$$

$$\text{CD:}\ \ \phi = 0\,,\ \ y = 0\,;\ \ \text{AD:}\ \ \varphi = \rho y\,,\ \ \phi = 0\,. \tag{1}$$

Here φ and ϕ are mutually conjugate functions harmonic within the region $z = x + ij$ and referred to κ, and Q is the filtration flow rate referred to κ, too. The problem is in determining the width l_1 and the height l_2 of the saltwater tongue penetrating into the freshwater stratum.

The complex-velocity region $w = dw/dz$ corresponding to boundary conditions (1) and to the flow diagram shown in Fig. 1 has the form of a circular triangle presented in Fig. 2a. We emphasize that in this case we have a bounded region with a zero apex angle D and apex angles A and B equal to $\pi(0.5 - \nu)$.

To solve the problem we introduce the auxiliary variable ζ and the functions: $z(\zeta)$ conformally mapping the upper half-plane ζ onto the region z (the correspondence of points is indicated in Fig. 2b) and the derivatives

$$Z = \frac{dz}{d\zeta}, \quad \Omega = \frac{d\omega}{d\zeta}. \tag{2}$$

Determining the characteristic exponents of the functions Z and Ω near the singular points, we find that they are linear combinations of two branches of the following Riemann function [8, 9]:

$$P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & -0.5 & -1 & 2 & \zeta \\ 0.5-\nu & \nu-1 & -1 & 2 & \end{Bmatrix} = \frac{1}{(a-\zeta)(1-\zeta)^{1-\nu}} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \nu & \zeta \\ 0.5-\nu & 0.5-\nu & \nu & \end{Bmatrix} = \frac{Y}{\Delta(\zeta)},$$

$$\Delta(\zeta) = (a-\zeta)(1-\zeta)^{1-\nu}. \tag{3}$$

From relation (3) it is seen that the point $\zeta = a$ is the ordinary point for the function Y; therefore, the Fuks linear differential equation corresponding to the Riemann symbol (3) takes the form

$$\zeta(1-\zeta) Y'' + (0.5 + \nu - (1 + 2\nu)\zeta) Y' - \nu^2 Y = 0. \tag{4}$$

As the two linear independent integrals of Eq. (4), we take the expressions

$$Y_1(\zeta) = F(\nu, \nu, 0.5 + \nu, 1 - \zeta), \quad Y_2(\zeta) = (1-\zeta)^{0.5-\nu} F(0.5, 0.5, 0.5 - \nu, 1 - \zeta), \tag{5}$$

forming a fundamental system of solutions in the vicinity of the point $\zeta = 1$. Here we have

$$F(\alpha, \beta, \gamma, x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)\,\Gamma(\gamma)}{n!\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma+n)} x^{n}, \tag{6}$$

where $\Gamma(x)$ is the Eulerian integral of the second kind [12].

The function performing conformal mapping of the upper half-plane ζ onto the region of the complex velocity w must be expressed by the ratio of linear combinations of Y_1 and Y_2. If we compose such combinations and use the correspondence of points A, B, and D on the ζ and w planes, we obtain

$$w = \rho \tan \pi\nu \left(1 - A\frac{Y_2(\zeta)}{Y_1(\zeta)}\right), \quad A = \frac{\tan \pi\nu \Gamma^2(1-\nu)}{\Gamma(1.5-\nu)\,\Gamma(0.5-\nu)}. \tag{7}$$

Taking relation (3) into account and allowing for expression (7), we find

$$Z = M \exp(\pi\nu i) \frac{Y_1(\zeta)}{\Delta(\zeta)}, \quad \Omega = M\rho \tan \pi\nu \frac{\exp(\pi\nu i)\, Y_1(\zeta) - iAY_2(\zeta)}{\Delta(\zeta)}, \quad M > 0. \tag{8}$$

We may assure ourselves that the functions (2) determined based on relation (8) satisfy boundary conditions (1) that have been formulated in terms of the functions mentioned and hence are the parametric solution of the initial boundary-value problem.

Writing the representations (8) for different portions of the boundary of the region ζ followed by integration over the entire contour of the auxiliary region ζ leads to closure of the contour of the region of motion z, which enabled us to control computations. As a result, we obtain the expressions

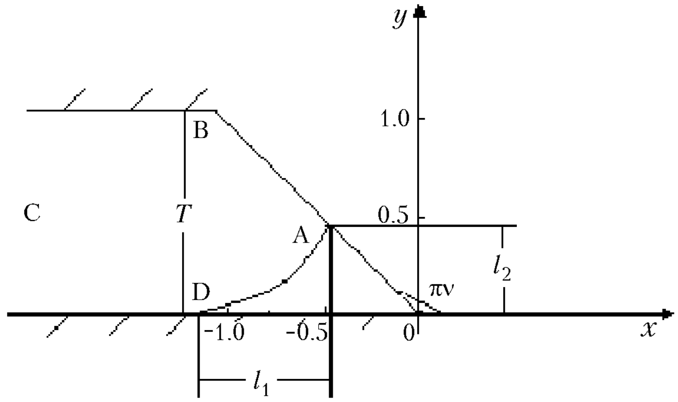

Fig. 3. Flow pattern calculated for $\nu = 0.75$, $T = 1.0$, $Q = 0.01$, and $\rho = 0.01$.

$$T = \frac{M\pi F\left(\nu, 0.5, \nu + 0.5, 1 - \frac{1}{a}\right)}{a^{\nu}\,(a-1)^{1-\nu}}, \quad Q = \frac{MB\rho\pi F\left(\nu, 0.5, 1, \frac{1}{a}\right)}{a^{\nu}\,(a-1)^{1-\nu}}; \tag{9}$$

$$l_1 = M\int_{-\infty}^{0} \frac{F\left(\nu, 0.5, \nu + 0.5, \frac{\zeta}{\zeta - 1}\right)}{(a-\zeta)\,(1-\zeta)}\,d\zeta, \quad l_2 = MB\int_{-\infty}^{0} \frac{F\left(\nu, 0.5, 1, \frac{1}{1-\zeta}\right)}{(a-\zeta)\,(1-\zeta)}\,d\zeta, \tag{10}$$

where $B = \sqrt{\pi}\,\Gamma(\nu + 0.5)/\Gamma(\nu)$. Also, we were able to exercise control using other expressions for the quantities T and Q:

$$T = l_2 - M\sin\pi\nu \int_0^1 \frac{F(\nu, \nu, \nu + 0.5, 1 - \zeta)}{\Delta(\zeta)}\,d\zeta,$$

$$Q = \frac{M\rho\sqrt{\pi}\,B}{\Gamma(1.5-\nu)\,\Gamma(\nu)} \int_0^1 \frac{\zeta^{0.5-\nu}\,F(0.5, 0.5, 1.5-\nu, \zeta)}{\Delta(\zeta)}\,d\zeta. \tag{11}$$

Case $0.5 \le \nu \le 1$. Figure 3 shows the filtration scheme corresponding to the case in question; the region of complex velocity for the scheme is presented in Fig. 2c where the dashed curve corresponds to $\nu = 0.5$. Since the condition $|w| = \rho$ must be observed on the boundary line, from Fig. 2c it is clear that, even for $\nu = 0.5$, the portions corresponding to the boundary line and to the seepage gap in the w plane just touch one another at point A and do not intersect as before. The latter leads to the fact that the complex-velocity region ceases to be bounded: when $\nu = 0.5$ we obtain a modular triangle [8, 9] all whose angles, including the apex angle B, are equal to zero. In the case where $0.5 \le \nu \le 1$ we have an unbounded circular triangle with apex angles A, B, and D equal to $\pi(\nu - 0.5)$, $-\pi(\nu - 0.5)$, and 0 respectively. Thus, the case $\nu = 0.5$ is the boundary and passage to the case $0.5 \le \nu \le 1$ is accompanied by the transformation of the complex-velocity region and by the reconstruction of the solution.

In the case in question the Riemann scheme corresponding to (3) takes the form

$$P\begin{Bmatrix} 0 & 1 & a & \infty & \\ 0 & -0.5 & -1 & 2 & \zeta \\ 0.5-\nu & \nu-1 & -1 & 2 & \end{Bmatrix} = \frac{1}{\Delta(\zeta)\,\zeta^{\nu-0.5}}\,P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & 0.5 & \zeta \\ \nu-0.5 & 0.5-\nu & 0.5 & \end{Bmatrix} = \frac{Y}{\Delta(\zeta)\,\zeta^{\nu-0.5}}.$$

The latter Riemann symbol corresponds to the linear differential equation of the Fuks class

$$\zeta\,(1-\zeta)\,Y'' + (1.5-\nu-2\zeta)\,Y' - 0.25Y = 0\,, \tag{12}$$

the fundamental system of whose solutions in the vicinity of the point $\zeta = 1$ has the form

$$Y_1(\zeta) = F(0.5, 0.5, 0.5+\nu, 1-\zeta)\,,\quad Y_2(\zeta) = (1-\zeta)^{0.5-\nu}\,F(1-\nu, 1-\nu, 0.5-\nu, 1-\zeta)\,. \tag{13}$$

Taking into account that the structure of the functions w, Z, and Ω does not change, we arrive at the previous expressions (9) for the quantities T and Q, (10) for l_1, and (11) for Q. As far as expressions (10) for l_2 and (11) for T are concerned, here they take the following form:

$$l_2 = MB\int_{-\infty}^{0}\frac{F\left(0.5, 1-\nu, 1, \dfrac{1}{1-\zeta}\right)}{\Delta(\zeta)\,(1-\zeta)^{0.5-\nu}\,(-\zeta)^{\nu-0.5}}\,d\zeta\,,\quad T = l_2 - M\sin\pi\nu\int_0^1\frac{F(0.5, 0.5, \nu+0.5, 1-\zeta)}{\Delta(\zeta)\,\zeta^{\nu-0.5}}\,d\zeta\,. \tag{14}$$

By virtue of the existing relation ([13], p. 133)

$$F(\alpha, \beta, \gamma, x) = (1-x)^{-\alpha}\,F\left(\alpha, \gamma-\beta, \gamma, \frac{x}{x-1}\right) \tag{15}$$

we will have

$$\left(\frac{\zeta}{\zeta-1}\right)^{0.5-\nu} F\left(0.5, 1-\nu, 1, \frac{1}{1-\zeta}\right) = F\left(0.5, \nu, 1, \frac{1}{1-\zeta}\right),$$

$$\zeta^{0.5-\nu}\,F(0.5, 0.5, \nu+0.5, 1-\zeta) = F(\nu, \nu, \nu+0.5, 1-\zeta)\,.$$

Consequently, in the expressions for l_2 and T, the right-hand sides of formulas (10), (11), and (14) coincide.

Thus, despite the transformation of the complex-velocity region, in which both the differential equations (4) and (12) themselves and their integrals (5) and (13) are different, the final dependences obtained for $0 < \nu < 0.5$ do remain true for the case $0.5 \le \nu \le 1$, too.

We note that the solution constructed for the functions Z and Ω (8) is quite analogous to the Polubarinova-Kochina solution of the problem on filtration of groundwater in a trapezoidal bridge in the absence of evaporation ([8], p.282, formulas (11.5)).

Solution of Problems for the Schemes of Mikhailov and Bear–Dagan. Polubarinova-Kochina Solution of the Problem on Filtration in an Infinitely Wide Bridge. In the case where $\nu = 0.5$ (Mikhailov scheme [3]) the hypergeometric functions appearing in relations (9)–(11) degenerate into a complete elliptic integrals of the first kind ([12], p. 919, formula 8.113.1)

$$F(0.5, 0.5, 1, k^2) = K(k^2)\,,\quad F(0.5, 0.5, 1, 1-k^2) = K(k'^2) = K'(k^2)\,.$$

Here we have $B = 1$ and expressions (9) take the form

$$T = \frac{M\pi K'\left(\dfrac{1}{a}\right)}{\sqrt{a\,(a-1)}}\,,\quad Q = \frac{M\rho K\left(\dfrac{1}{a}\right)}{\sqrt{a\,(a-1)}}\,.$$

The last formulas coincide with relations (9) from [4]. In the case in question the region of flow becomes the mirror image of the region of groundwater motion in an infinitely wide bridge. The existing solution of this problem, which was obtained for the first time by P. Ya. Polubarinova-Kochina ([8], p. 275, formulas (10.44)–(10.46)), follows from expressions (14) upon the replacement of T and l_2 in them by H_1 and H_0 respectively.

When $\nu = 1$ (Bear–Dagan scheme [1]) the hypergeometric functions degenerate into elementary ones [12] (p. 1055, formulas (9.121.7) and (9.121.25)):

$$F(1, 0.5, 1.5, k^2) = \frac{\ln\left(\dfrac{1+k}{1-k}\right)}{2k}, \quad F(1, 0.5, 1, k^2) = \frac{1}{k'}.$$

In this case we have $B = 0.5\pi$ and formulas (9) take the form

$$T = \frac{M\pi\sqrt{a}\,\ln(\sqrt{a} + \sqrt{a-1})}{\sqrt{a-1}}, \quad Q = \frac{M\pi\rho\sqrt{a}}{2\sqrt{a-1}}.$$

These relations coincide with expressions (6.4) from [4].

Transformation of Formulas to a Form Convenient for Computations. The representations (9)–(11) contain two unknown constants M and a ($1 < a < \infty$). The *Q/T* ratio is used for determination of the parameter of mapping a: from expressions (9) we obtain

$$\frac{BF\left(\nu, 0.5, 1, \dfrac{1}{a}\right)}{F\left(\nu, 0.5, \nu + 0.5, 1 - \dfrac{1}{a}\right)} = \frac{Q}{\rho T}. \tag{16}$$

The constant M is preliminarily eliminated from all Eqs. (10) and (11) by means of the first of relations (9) (it fixes the thickness T of the water-bearing layer). We note that relation (16) regulates prescription of the parameters Q, T, ρ, and ν and consequently applicability of the flow diagram adopted: this means that the solution of Eq. (16), just as before [4–7], does not exist for all combinations of these parameters.

The main computational difficulty of the problem is that the integrands appearing in relations (10) and (11) are infinite on the limits of integration. Furthermore, in finding the quantity l_1 from formula (10), the parameters of the hypergeometric function obey the condition $\gamma = \alpha + B$, so that the traditional formulas lose their meaning in analytical extension of this function to the interval $(-\infty, 0)$.

We introduce the notation $\alpha = 1/a$ ($0 < \alpha < 1$), $\alpha_1 = 1 - \alpha$, and $C = M\alpha$ for convenience of computations. Next, taking into account that ([13], p. 72)

$$\int_0^1 x^{\lambda}(1-x)^{\mu}(1-tx)^{r}\,dx = \frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}\,F(r, \lambda+1, \mu+\lambda+2, t),$$

and using relation (15), we apply the formula to analytical extension of the hypergeometric function in our particular case ([13], p. 177). As a result, we arrive at calculated dependences corresponding to expressions (9)–(11):

$$T = \frac{C\pi F(\nu, 0.5, \nu+0.5, \alpha_1)}{\alpha_1^{1-\nu}}, \quad Q = \frac{CB\rho F(\nu, 0.5, 1, \alpha)}{\alpha_1^{1-\nu}}; \tag{17}$$

$$l_1 = CB\int_0^1 (1-\alpha_1 x)^{-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+0.5)\,\Gamma(n+\nu)}{\Gamma(n+0.5+\nu)\,(n!)^2}\,S_n(x)\,dx,$$

$$S_n(x) = [2\psi(n+1) - \psi(n+\nu) - \psi(n+0.5) - \ln(1-x)]\,(1-x)^n; \tag{18}$$

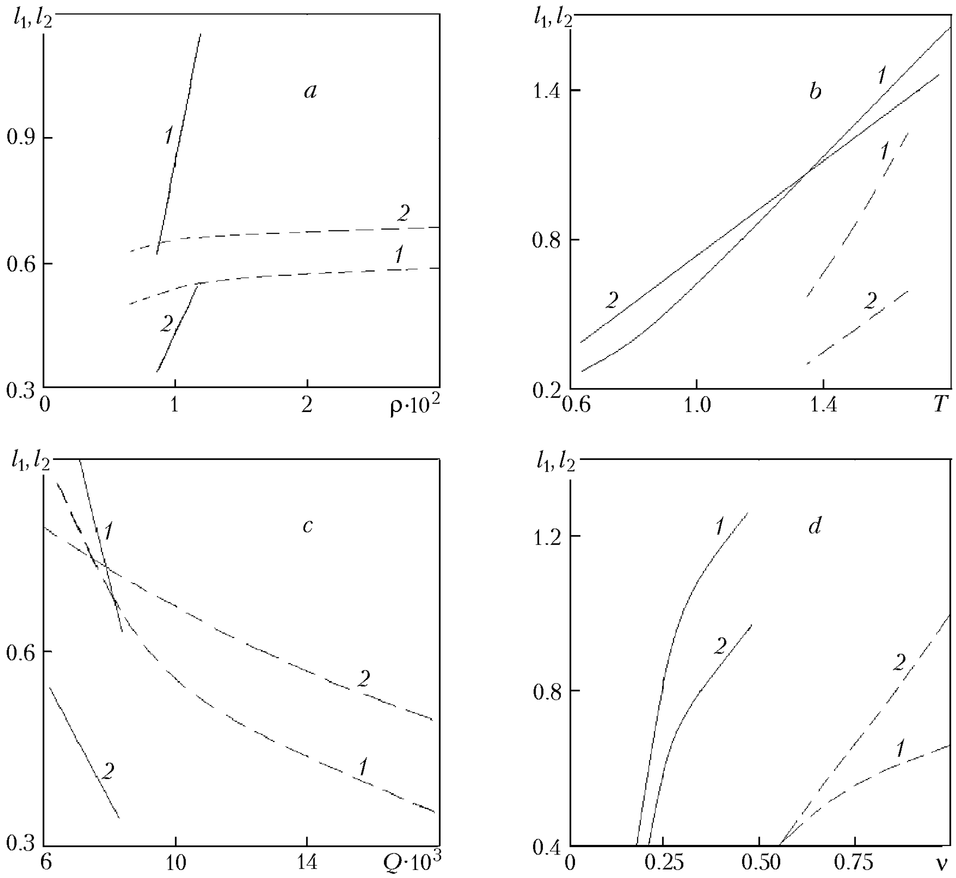

Fig. 4. Dependence of l_1 and l_2 on ρ (a), T (b), Q (c), and ν (d) (T = 1.5, Q = 0.007, and ρ = 0.01 for ν = 0.25 and T = 1.0, Q = 0.01, and ρ = 0.01 for ν = 0.75): 1) l_1 and l_2. Solid curves, flow diagram for $0 < \nu < 0.5$, dashed curves, diagram for $0.5 \le \nu \le 1$.

$$l_2 = \frac{CB}{\sqrt{\pi}\,\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n+0.5)\,\Gamma(n+\nu)}{(n!)^2\,(n+1)} F(1, 1, n+2, \alpha_1)\,,$$

$$T = l_2 - \frac{CB \sin \pi\nu}{\sqrt{\pi}\,\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma^2(n+\nu)}{(n!)\,(n+\nu)\,\Gamma(n+0.5+\nu)} F(1, 1, n+1+\nu, \alpha)\,,$$

$$Q = \frac{CB\rho}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+0.5)}{(n!)\,(n+0.5)} F(1, n+0.5-\nu, n+0.5, \alpha)\,. \tag{19}$$

Here we have $\psi(x) = d \ln \Gamma(x+1)/dx$ ([13], p. 953).

Calculation of the Flow Diagram and Analysis of the Numerical Results. Figures 1 and 3 show the flow patterns calculated for ν = 0.25, T = 1.5, Q = 0.007, and ρ = 0.01 and ν = 0.75, T = 1, Q = 0.01, and ρ = 0.01 respectively. Figure 4 plots the quantities l_1 (curves 1) and l_2 (curves 2) as functions of ν for $0 < \nu < 0.5$ as the solid curves and for $0.5 \le \nu \le 1$ as the dashed curves. An analysis of the plots suggests the following.

First of all we note that the increase in the thickness of the water-beating layer, the density of saltwater, and the angle of inclination of the littoral part of the sea-floor zone and the decrease in the flow rate increase the dimensions of the saltwater tongue. Thus, when $\nu = 0.25$ the quantities l_1 and l_2 increase by 791 and 987% respectively, as the parameters ρ, Q, and T decrease 1.24 times. The relative dimensions of the tongue may be quite large: we have $l_1 = 1.9076$ and $l_2 = 1.5715$ for $\nu = 0.25$, $\rho = 0.01$, $Q = 0.01$, and $T = 1.921$, i.e., the tongue height and width may attain 82 and 99% of the stratum thickness.

A substantial influence on the degree of intrusion is exerted by the size of the angle ν; the changes are the most significant for low values of ν, ρ, and T, ν values close to 0.5, and high values of Q. Thus, we obtain $l_1 = 0.0262$ and $l_2 = 0.0107$ for $T = 1,5$, $Q = 0.007$, $\rho = 0.0089$, and $\nu = 0.1667$ and $l_1 = 1.6670$ and $l_2 = 1.0184$ for $\nu = 0.49$, i.e., a change of 3 times in the angle ν leads to an increase of 63.6 and 95.2 times in the dimensions of the tongue respectively. Also, it is noteworthy that in the case where $0 < \nu < 0.5$ the width of the tongue exceeds its height 1.53 to 2.48 times; the inequality $l_2/(\sin \pi\nu) < l_1$ holds, too, as a rule, i.e., the length of the portion OA is smaller than the tongue width. When $0.5 \le \nu \le 1$, conversely, the length of the portion OA is larger than the width l_1 almost without exception. For low values of the parameters ρ and T and high Q values, the height of the tongue l_2 exceeds its width now; the difference may attain 36%: when $T = 0.658$ we have $l_1 = 0.2619$, $l_2 = 0.3571$, and $l_2/(\sin \pi\nu) = 0.5051$, so that $\mathrm{OA} \approx 1.9 l_1$.

It is remarkable that in the case of low values of the parameters ν, ρ, and T and high values of ν and Q the approximate equality $l_2/(\sin \pi\nu) \approx l_1$ holds, as before [4–7], i.e., the lengths of the portions OD and OA are nearly coincident.

From the plot given in Fig. 4a it is clear that the dependences of the tongue dimensions l_1 and l_2 on the parameter ρ are nearly linear for $0 < \nu < 0.5$, and they are qualitatively similar for $0.5 \le \nu \le 1$. It is seen that the quantity l_2 is nearly linearly dependent on the other parameters, too: on T (Fig. 4b), Q for $0 < \nu < 0.5$ (Fig. 4c), and ν for $0.5 \le \nu \le 1$ (Fig. 4d).

Conclusions. We have found the exact analytical solution of the problem on intrusion of saltwater into freshwater strata in the most general case where the littoral part of the sea-floor zone (slope) makes an arbitrary angle with the horizon. It has been established by means of numerical calculations that the increase in the thickness of the water-bearing layer, the density of saltwater, and the angle of inclination of the littoral part of the sea-floor zone and the decrease in the flow rate lead to a growth of the saltwater tongue penetrating into the freshwater stratum.

We give the solutions for the case of flow where the angle of inclination is equal to a right angle (Mikhailov scheme), within whose framework the well-known Polubarinova-Kochina solution of the problem on groundwater filtration in an infinitely wide bridge is noted, and for the limiting case of motion where the angle of inclination is equal to a right angle (Bear–Dagan scheme).

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NOTATION

a, unknown affix (i.e., image) of point C on the auxiliary-variable plane; k, modulus of elliptic integrals of the first kind; k', additional modulus; K and K', complete elliptic integrals of the first kind of the modulus k and the additional modulus k'; l_1 and l_2, width and height of the saltwater tongue, m; M, scale constant of modeling; n, summation index; P, Riemann symbol corresponding to two solutions of the Fuks equations, that contain singular points and the exponents of the functions Z and Ω sought in them; Q, water flow rate; r and t, parameters of the hypergeometric function; T, stratum thickness, m; x and y, abscissa and ordinate of a point of the flow region; Y, solution of the Fuks equation; Y_1 and Y_2, first and second integrals of the Fuks equation; z, complex coordinate of a point of the flow region; Z, sought function; w, complex flow velocity; ζ, auxiliary parametric variable; κ, filtration factor; ρ_1 and ρ_2, densities of freshwater and saltwater; ν, angle of inclination of the littoral part of the sea-floor zone (slope); φ, velocity potential; ϕ, stream function; ψ, logarithmic derivative of the Eulerian integral of the second kind; ω, complex flow potential.

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